M.Sc. - II (Mathematics) (NEP Pattern) Semester-III

03NEPMATH01 - Major - Complex Analysis

P. Pages : 2 **GUG/S/25/16013**

Time : Three Hours Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) If G is open and connected and $f: G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G, then prove that f is constant. **8**

b) Let G & Ω be open subsets of $\mathbb{C}$. Suppose that $f: G \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ are continuous functions such that f(G) C Ω & g(f(z)) = z for all z in G. If g is differentiable & $g'(z) \neq 0$, then prove that f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, prove that f is analytic. **8**

OR

c) Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u: G \to \mathbb{R}$ is a harmonic function then prove that u has a harmonic conjugate. **8**

d) If $\Sigma a_n (z-a)^n$ is a given power series with radius of convergence R, then prove that $R = \lim \left| \dfrac{a_n}{a_{n+1}} \right|$ if this limit exists. **8**

UNIT – II

2. a) Let $f: G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G (r > 0)$. If $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, then prove that $f(z) = \dfrac{1}{2\pi i} \int_\gamma \dfrac{f(\omega)}{\omega - z} d\omega$ for $|z-a| < r$. **8**

b) If p(z) is a nonconstant polynomial then prove that there is a complex number a with P(a) = 0. **8**

OR

c) If G is a region & $f: G \to \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \ge |f(z)|$ for all z in G, then prove that f is constant. **8**

d) If $\gamma: [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then prove that $\dfrac{1}{2\pi i} \int_\gamma \dfrac{dz}{z-a}$ is an integer. **8**

UNIT – III

3. a) Let G be a region & let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G; then prove that f is analytic in G. **8**

b) Let G be a region and suppose that f is a nonconstant analytic function on G. Then prove that for any open set U in G, f(U) is open. **8**

OR

c) Show that for a > 1, $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$ **8**

d) Suppose f & g are meromorphic in a neighborhood of $\bar{B}(a;R)$ with no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$. If $Z_f, Z_g (P_f, P_g)$ are the number of zeros (poles) of f & g inside γ counted according to their multiplicities & if $|f(z)+g(z)| < |f(z)|+|g(z)|$ on γ, then prove that $Z_f - P_f = Z_g - P_g$. **8**

UNIT – IV

4. a) Prove that a differentiable function f on [a, b] is convex iff f' is increasing. **8**

b) If f is analytic in a region G and a is a point in G with $|f(a)| \ge |f(z)|$ for all z in G then prove that f must be a constant function. **8**

OR

c) Let $D = \{z : |z| < 1|\}$ and suppose f is analytic on D with **8**

a) $|f(z)| \le 1$ for z in D

b) f(0) = 0

Then prove that $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all z in the disk D. Moreover, if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \ne 0$ then prove that there is a constant $C, |C| = 1$, such that $f(\omega) = c\,\omega$ for all ω in D.

d) Let $a \ge \frac{1}{2}$ and put $G = \left\{ z : |\arg z| < \frac{\pi}{2a} \right\}$. Suppose that f is analytic on G & there is a constant M such that $\lim_{z\to\omega} \sup |f(z)| \le M$ for all ω in ∂G. If there are positive constants p and b < a such that $|f(z)| \le P \exp(|z|^b)$ for all z with $|z|$ sufficiently large, then prove that $|f(z)| \le M$ for all z in G. **8**

5. a) Define differentiable function and analytic function. **4**

b) Define: i) Path ii) Smooth path **4**

c) Define: i) Meromorphic function ii) Essential singularity **4**

d) Define convex function and convex set. **4**

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